2022 CAC High School Math Contest - Open Competition

1. Expand $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^4$

A) 1 + i B) 1 C) 1 - i D) -1 E) None of the above

Use the linear approximation $(1+x)^k \approx 1 + kx$, as specified.

2) Estimate $\sqrt[3]{1.012}$.

A) 1.005 B) 1.004 C) 1.04 D) 1.05 E) None of the above

3) Equating the numerators in the process of partial fraction decomposition, your friend has obtained the algebraic expression $8x - 5 = A(3x - 7) + B(x + 3)$. From this equation, what can you say about the original expression?

A) The original expression has repeated linear factors in the denominator.
B) The denominator of the original expression is $3x^2 + 2x - 21$.
C) The original expression has a non-reducible quadratic factor in the denominator.
D) The numerator of the original expression is $3x^2 + 2x - 21$.
E) None of the above

4) Find the extrema of the function on the given interval, and say where they occur.

$\sin x + \cos x$, $0 \le x \le 2\pi$

A) local maxima: 1 at x = 2π and $\sqrt{2}$ at x = $\frac{\pi}{4}$;

local minima: 1 at x = 0 and $-\sqrt{2}$ at x = $\frac{5\pi}{4}$

B) local maxima: 1 at x = 0 and $-\sqrt{2}$ at x = $\frac{7\pi}{4}$;

local minima: 1 at x = 2π and $\sqrt{2}$ at x = $\frac{\pi}{4}$

C) local maxima: 1 at x = 0 and $-\sqrt{2}$ at x = $\frac{5\pi}{4}$;

local minima: 1 at x = 2π and $\sqrt{2}$ at x = $\frac{\pi}{4}$

D) local maxima: 1 at x = 2π and $\sqrt{2}$ at x = $\frac{\pi}{4}$;

local minima: 1 at x = 0 and $-\sqrt{2}$ at x = $\frac{7\pi}{4}$

E) None of the above

Solve.

5) In how many distinguishable ways can the letters of the word STARTER be arranged?

A) 1260 B) 2520 C) 630 D) 5040 E) None of the above

Express the following logarithm as specified.

6) ln $\sqrt{40.5}$ in terms of ln 3 and ln 2

A) $\frac{4 \ln 3 - \ln 2}{2}$ B) $\frac{4 \ln 3}{2}$ C) $\frac{4 \ln 3 + \ln 2}{2}$ D) 4 ln 3 E) None of the above

Use the appropriate addition formula to find the exact value of the expression.

7) tan $\left(-\frac{7\pi}{12}\right)$

A) -2 - $\sqrt{3}$ B) $\frac{2 - \sqrt{3}}{4}$ C) $\frac{2 + \sqrt{3}}{4}$ D) 2 + $\sqrt{3}$ E) None of the above

Provide an appropriate response.

8) Find the absolute maximum and minimum values of f(x) = 2x - e^x on [0, 1].

A) Maximum = 0 at x = ln 2, minimum = 1 at x = 0
B) Maximum = ln 4 - 2 at x = ln 2, minimum = -1 at x = 0
C) Maximum = ln 4 - 2 at x = ln 2, minimum = 1 at x = 0
D) Maximum = ln 2 - 2 at x = ln 2, minimum = -1 at x = 0
E) None of the above

9) The equation gives the position s = f(t) of a body moving on a coordinate line (s in meters, t in seconds).
s = 1 + 11 cos t

Find the body's jerk at time t = π/3 sec.

A) - $\frac{11}{2}$ m/sec^3 B) $\frac{11}{2}$ m/sec^3 C) $\frac{11\sqrt{3}}{2}$ m/sec^3 D) - $\frac{11\sqrt{3}}{2}$ m/sec^3

E) None of the above

10) Find equations of all tangents to the curve f(x) = $\frac{1}{x + 16}$ that have slope -1.

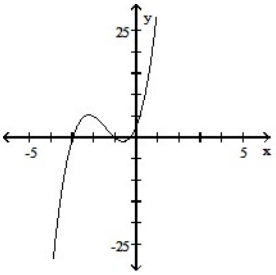
A) y = -x - 14 B) y = -x - 14, y = -x – 18

C) y = -x + 18 D) y = -x + 18, y = -x – 14 E) None of the above

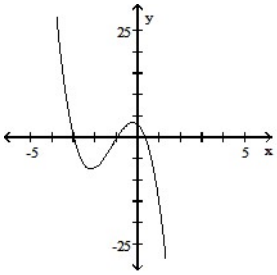
11) Graph the polynomial function and find the zeros.

f(x) = $3x^3$ + $7x^2$ - 7x - 3

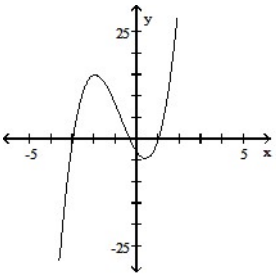
A) 1, $\frac{1}{3}$, 3;



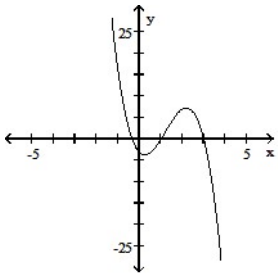
B) 1, $\frac{-1}{3}$, 3



C) 1, - $\frac{1}{3}$, -3;



D) 1, $\frac{1}{3}$, -3



E) None of the above

12) Find the most general antiderivative.

$$\int \frac{x\sqrt{x} + \sqrt{x}}{x^2}\,dx$$

A) C

B) - $\frac{\sqrt{x}}{2}$ - $\frac{3\sqrt{x}}{2}$ + C

C) $2\sqrt{x}$ - $\frac{2}{\sqrt{x}}$ + C

D) $\frac{2}{\sqrt{x}}$ - $2\sqrt{x}$ + C

E) None of the above

13) Find y''' if y = 3x sin x.

A) y''' = - 3x cos x + 9 sin x

B) y''' = 3x cos x + 9 sin x

C) y''' = - 3x cos x - 9 sin x

D) y''' = 6 cos x - 3x sin x

E) None of the above

14) A function f(x), a point c, the limit of f(x) as x approaches c, and a positive number ε is given. Find a number $\delta > 0$ such that for all x, $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

f(x) = -7x + 10, L = -18, c = 4, and ε = 0.01

A) δ = -0.0025 B) δ = 0.005714 C) δ = 0.002857 D) δ = 0.001429
E) None of the above

15) Solve $x^5 + x^2 \geq 5x^3 + 5$

A) $(-\infty, -\sqrt{5}\,] \cup [-1, \sqrt{5}\,]$ B) $[-5, \infty)$

C) $[-\sqrt{5}, \sqrt{5}]$ D) $[-\sqrt{5}, -1] \cup [\sqrt{5}, \infty)$

E) None of the above

16) A man earned $2500 the first year he worked. If he received a raise of $500 at the end of each year, what was his salary during the 15th year?

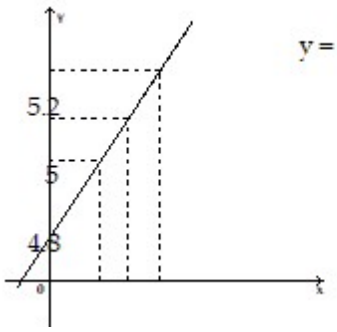
A) $2500 B) $7000 C) $9500 D) $10,000 E) None of the above

17) Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point (2, 4).

A) a = 1, b = 4 B) a = -1, b = 2 C) a = 1, b = 2 D) a = -1, b = 4

E) None of the above

Use the graph to find a δ > 0 such that for all x, $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
18)



$y = \frac{3}{2}x + 2$

1.9 2 2.1
Not to scale

$f(x) = \frac{3}{2}x + 2$

c = 2
L = 5
ε = 0.2

A) δ = 0.2 B) δ = 3 C) δ = 0.1 D) δ = -0.2 E) None of the above

19) Express the complex number in trigonometric form.

$-4\sqrt{3} - 4i$ Express your answer in radians.

A) $4\sqrt{3}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ B) $8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

C) $4\sqrt{3}\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right)$ D) $8\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

E) None of the above

20) Find $\frac{dy}{dx}$. $\ln 8xy = e^{x+y}$

A) $\frac{y}{x}$ B) $\frac{xye^{x+y} - y}{x - xye^{x+y}}$ C) $\frac{e^{x+y}}{e^{8x}}$ D) $\frac{2xye^{x+y}}{x + y}$

E) None of the above

21) Express as a single logarithm and, if possible, simplify.
$\ln (6 \sec \theta) + \ln (2 \cos \theta)$

A) ln(3) B) ln (12 cot θ)

C) ln(12) D) ln (6 sec θ + 2 cos θ) E) None of the above

22) Use the discriminant to determine whether the graph of the equation is an ellipse (or circle), a hyperbola, or a parabola.

$8x^2 - 7xy + 7y^2 - 20 = 0$

A) Hyperbola B) Ellipse or circle C) Parabola D) All the above

E) None of the above

23) For the given angle of rotation and coordinates of a point in the xy-coordinate system, find the coordinates of the point in the x'y'-coordinate system. θ = 60°, (-7, 0)

A) $\left(-\frac{7}{2}, \frac{-7\sqrt{3}}{2}\right)$ B) $\left(-\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)$ C) $\left(\frac{7\sqrt{3}}{2}, -\frac{7}{2}\right)$ D) $\left(\frac{-7\sqrt{3}}{2}, \frac{7}{2}\right)$

E) None of the above

24) A projectile is fired from ground level with an initial velocity of 400 feet per second at an angle of 45° with the horizontal. Find parametric equations that gives the position of the projectile at time t, in seconds.

A) $x = (400 \cos 45°)t$, $y = (400 \sin 45°)t + 16t^2$
B) $x = (400 \sin 45°)t$, $y = (400 \cos 45°)t + 16t^2$
C) $x = (400 \cos 45°)t$, $y = (400 \sin 45°)t - 16t^2$
D) $x = (400 \sin 45°)t$, $y = (400 \cos 45°)t - 16t^2$
E) None of the above

Provide an appropriate response.

25) If $\frac{x+y}{x-y} = x^2 + y^2$ and dx/dt = 12, then what is dy/dt when x = 1 and y = 0?

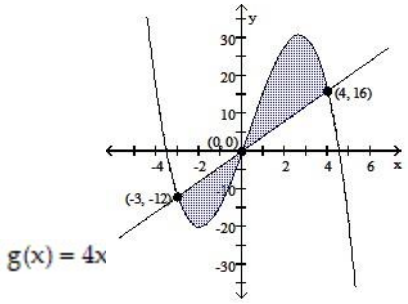
A) -12 B) $-\frac{1}{12}$ C) 12 D) $\frac{1}{12}$ E) None of the above

26) Find the limit. $\lim_{x \to \infty} \sec^{-1} x$

A) -∞ B) 0 C) $-\frac{\pi}{2}$ D) $\frac{\pi}{2}$ E) None of the above

27) Find the area of the shaded region.

$f(x) = -x^3 + x^2 + 16x$



A) $\frac{343}{12}$ B) $\frac{1153}{12}$ C) $-\frac{343}{12}$ D) $\frac{937}{12}$ E) None of the above

28) Suppose the velocity of a body moving along the s-axis is $\frac{ds}{dt}$ = 9.8t - 9.

Is it necessary to know the initial position of the body to find the body's displacement over some time interval? Justify your answer.

A) No, the initial position is necessary to find the curve s= f(t) but not necessary to find the displacement. The initial position determines the integration constant. When finding the displacement the integration constant is subtracted out.

B) No, displacement has nothing to do with the position of the body.

C) Yes, integration is not possible without knowing the initial position.

D) Yes, knowing the initial position is the only way to find the exact positions at the beginning and end of the time interval. Those positions are needed to find the displacement.

E) None of the above

State the domain and range of the function.

29) f(x) = $\sqrt{16 + 5^{-x}}$

A) domain: $(-\infty, \infty)$; range: (0, 4) B) domain: $(-\infty, \infty)$; range: (4, ∞)

C) domain: (0, ∞); range: $(-\infty, \infty)$ D) domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

E) None of the above

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1) D
2) B
3) B
4) A
5) A
6) A
7) D
8) B
9) C
10) B

11) C
12) C
13) C
14) D
15) D
16) C
17) D
18) C
19) B
20) B

21) C
22) B
23) B
24) C
25) C
26) D
27) D
28) A
29) B